

ON THE EXISTENCE OF BIBUNDLES

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ABSTRACT. We consider the existence of bibundles, in other words locally trivial principal G spaces with commuting left and right G actions. We show that their existence is closely related to the structure of the group $\text{Out}(G)$ of outer automorphisms of G . We also develop a classifying theory for bibundles. The theory is developed in full generality for (H, G) bibundles for a crossed-module (H, G) and we show with examples the close links with loop group bundles.

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1. INTRODUCTION

There has been interest recently in gerbes that have non-abelian band [1] particularly for applications to string theory. In the literature that has developed it is notable that it seems to be difficult to find concrete examples which are not closely related to abelian gerbes. A closer examination suggests that the problem centres

2010 *Mathematics Subject Classification.* 55R65, 53C08, 18D05.

Key words and phrases. bibundles, crossed modules, 2-groups, gerbes.

The first author acknowledges the support of the Australian Research Council. The second author acknowledges the support of an Australian Postgraduate Research Award. The third author was supported by the Engineering and Physical Sciences Research Council [grant number EP/I010610/1]. Finnur Larusson and Klaas Landsman are thanked for useful comments made after talks on this topic. We thank the referee for many useful comments.

around the need, when defining a gerbe, to be able to form a product of principal G bundles in such a way as to yield another principal G bundle (rather than a $G \times G$ bundle). A similar problem arises in module theory when one wants to take a tensor product of modules of a non-commutative ring R . In this case one is led to consider bimodules, i.e. modules over the ground ring with commuting left and right actions of R . By analogy, we are naturally led to study not just principal bundles but so-called ‘bibundles’. These are fibrings that are simultaneously left and right principal G bundles in such a way that the left and right G actions commute.

However, the existence of such objects is more problematic. To see why, consider the fibre of such a bibundle. In the case of a G bundle the fibre is a right G space and there is only one of these up to isomorphism. In the case of bibundles the fibre is a G *bispace* and now there are different isomorphism classes labelled by $\text{Out}(G)$, the group of outer automorphisms of G . We dwell in some apparently pedantic detail on the structure of G bispaces. This effort however is rewarded by making many constructions for bibundles immediate.

Fundamental to our approach is the idea of the *type* of a bispace or bibundle. In the case of bispaces the type of a bispace is an element of $\text{Out}(G)$ which classifies its isomorphism class. In the case of a bibundle $P \rightarrow M$ it is a map from M to $\text{Out}(G)$ whose value at $m \in M$ classifies the isomorphism class of the bispace which is the fibre of P over m . We call the map which associates to a bibundle the type of each of its fibres the *type map*. It forms part of an exact sequence of pointed sets

$$\pi_0 \mathbf{Bun}_{Z(G)}(M) \xrightarrow{\iota} \pi_0 \mathbf{Bibun}_G(M) \xrightarrow{\text{Type}} \text{Map}(M, \text{Out}(G))$$

where $Z(G)$ is the center of G , and $\pi_0 \mathbf{Bun}_{Z(G)}(M)$ is the pointed set of isomorphism classes of $Z(G)$ -bundles over M . The message that follows from the exactness of this sequence of pointed sets is that for genuinely non-abelian bibundles to exist we need $\text{Out}(G)$ to be large. In the case that G is simple and simply-connected $\text{Out}(G)$ is well known to be the (small) finite group of automorphisms of the Dynkin diagram of G . More interesting examples arise when G is the group ΩK of based loops in a compact group K whose outer automorphism group has large subgroups such as K itself.

In summary then we start in Section 2 with a detailed discussion of bispaces. As we will see, bispaces are a partial ‘categorification’ of the notion of G -spaces in which the structure group is replaced by a certain kind of groupoid — a so-called ‘2-group’ or crossed module. For simplicity in this introduction we have only considered G bispaces, which correspond to a restricted class of 2-groups. To obtain a more flexible theory we will need to discuss the more general case of (H, G) bispaces for a crossed module (H, G) . In Section 3 we consider the case of (H, G) bibundles and as well as explaining the exact sequence above we consider the classifying theory of bibundles. Finally in the conclusion we indicate briefly some results on the more complicated case of (H, G) bibundle gerbes.

2. BISPACES

Let G be a topological group and let X be a space. If G acts freely and transitively on both the left and the right of X and these actions commute, we call X a G *bispace*. We will very often be interested in a smooth version of this notion where G is a Lie group, X is a manifold and both of the actions of G on X are smooth — for convenience we will also call these objects G bispaces. However, it turns out that it

is more natural to consider a more general notion, that of an (H, G) bispace, where (H, G) is a ‘crossed module’. In the next subsection we will give some motivation for this, building up to the definition of (H, G) bispace (see Definition 2.2 below).

2.1. (H, G) bispaces. First consider a G bispace X and, following [1, 4], define a map

$$\psi: X \rightarrow \text{Aut}(G)$$

by $xg = \psi(x)(g)x$. Note that $\psi(x)$ is indeed in $\text{Aut}(G)$ because we have $\psi(x)(gh)x = xgh = \psi(x)(g)xh = \psi(x)(g)\psi(x)(h)x$. We call ψ the *structure map* of X . We have

$$\psi(xg)(h)xg = xgh = x(ghg^{-1})g = \psi(x)(ghg^{-1})xg = \psi(x) \text{Ad}(g)(h)xg$$

so that $\psi(xg) = \psi(x) \circ \text{Ad}(g)$. Thus ψ is right G equivariant if we consider X to be a right G space and $\text{Aut}(G)$ a right G space under the adjoint action. The data of the right G space X together with the equivariant map $\psi: X \rightarrow \text{Aut}(G)$ is sufficient to recover the bispace X , more precisely we have the following lemma from [4].

Lemma 2.1 ([4] Lemme 2.5). *The structure map of a bispace gives rise to an equivalence between*

- (1) G bispaces X
- (2) Pairs (X, ψ) consisting of a right G space X and an equivariant map $\psi: X \rightarrow \text{Aut}(G)$.

A slightly more general idea would be to choose a subgroup $H \subset \text{Aut}(G)$ containing $\text{Ad}(G)$ and require that the structure map ψ take values in H , in other words $\psi(x) \in H$ for all $x \in X$. We will take one step beyond this. Recall (see for instance [2, 4]) that a *crossed module* is a pair of topological groups (H, G) together with homomorphisms

$$G \xrightarrow{t} H \xrightarrow{\alpha} \text{Aut}(G)$$

satisfying the following two conditions:

- (1) t is H -equivariant for the action of H on G defined by α and the adjoint action Ad_H of H on G , that is $t(\alpha(h)(g)) = ht(g)h^{-1}$,
- (2) the action of G on itself induced by t is the adjoint action of G on itself, i.e. $\alpha \circ t = \text{Ad}_G$.

Note that we have the following example.

Example 2.1. If $H \subset \text{Aut}(G)$ is a subgroup containing $\text{Ad}(G)$ then

$$G \xrightarrow{\text{Ad}} H \hookrightarrow \text{Aut}(G).$$

is a crossed-module.

A *morphism* of crossed modules $(H, G) \rightarrow (H', G')$ consists of a pair of homomorphisms $u: H \rightarrow H'$ and $v: G \rightarrow G'$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{v} & G' \\ t \downarrow & & \downarrow t' \\ H & \xrightarrow{u} & H' \end{array}$$

commutes and the equivariance condition $v(\alpha(h)(g)) = \alpha'(u(h))(v(g))$ is satisfied.

Two easy consequences of the definition of crossed module are the following: $G_1 = \ker(t) \subset Z(G)$, the centre of G (hence $\ker(t)$ is abelian), and also $t(G) \subset H$ is normal. Therefore we have exact sequences of groups

$$1 \rightarrow G_1 \rightarrow G \rightarrow G/G_1 \rightarrow 1$$

and

$$1 \rightarrow t(G) \rightarrow H \rightarrow H/t(G) \rightarrow 1.$$

Throughout we will assume that the projections $G \rightarrow G/G_1$ and $H \rightarrow H/t(G)$ admit local sections (if $H \rightarrow G$ is a crossed module of Lie groups this is of course immediate, unless one is dealing with some classes of infinite dimensional Lie groups). We will sometimes adopt the notation $G \rightarrow H$ for a crossed module. The following definition appears in [4].

Definition 2.2 ([4] page 432). Let (H, G) be a crossed module. An (H, G) *bispace* is a pair (X, ψ) consisting of a right G space X and an equivariant map $\psi: X \rightarrow H$.

By equivariant we mean that $\psi(xg) = \psi(x)t(g)$. We shall often write bispace instead of (H, G) bispace when the context is clear and call the map ψ the *structure map* of X . Note that our definition of (H, G) bispace is different to that in [1] where such a thing, in our notation, is a left H -space with an equivariant map to $\text{Aut}(G)$.

Example 2.2. If G is a topological group then there is a canonical crossed module

$$G \xrightarrow{\text{Ad}} \text{Aut}(G) \xrightarrow{\text{id}} \text{Aut}(G).$$

It is straightforward from Lemma 2.1 to see that an $(\text{Aut}(G), G)$ bispace is the same thing as a G bispace.

If X is an (H, G) bispace with structure map $\psi: X \rightarrow H$ then $\alpha \circ \psi: X \rightarrow \text{Aut}(G)$ is equivariant and we have the following lemma.

Lemma 2.3. *If (X, ψ) is an (H, G) bispace then $(X, \alpha \circ \psi)$ is a G -bispace (via the correspondence of Lemma 2.1).*

This means that an (H, G) bispace has a left action of G defined by

$$(2.1) \quad gx = x((\alpha \circ \psi)(x))^{-1}(g)$$

which we use in the future, often without comment. Note however that $\psi(gx) = t(g)\psi(x)$.

If $h \in H$ denote by $[h]$ the coset in the quotient group $H/t(G)$. By equivariance ψ defines a unique element $\phi = [\psi(x)] \in H/t(G)$ which we call the *type* of X . If 1 denotes a point we have the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & H \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\phi} & H/t(G) \end{array}$$

where $\phi(1) = \phi$. We write $\text{Type}(X)$ for the type of X .

In the case of a G bispace X we have that $\text{Type}(X) \in \text{Aut}(G)/\text{Ad}(G) = \text{Out}(G)$, the group of outer automorphisms of G . We have seen (Lemma 2.3) that any (H, G) bispace is also a G bispace and hence has a type in $\text{Out}(G)$. This is the image of the type in $H/t(G)$ under the homomorphism

$$H/t(G) \rightarrow \text{Out}(G)$$

induced by α .

Example 2.3. Choose an element $\xi \in H$ and let $X = G$, considered as a right G -space under group multiplication. Define a structure map $\psi: G \rightarrow H$ by $\psi(x) = \xi t(x)$. The induced bispace left action on X is given by

$$k \star x = x(\alpha\psi(x))^{-1}(k) = x(\alpha(\xi t(x)))^{-1}(k) = x(\alpha(\xi) \text{Ad}(x))^{-1}(k).$$

Denote this (H, G) bispace by $T(\xi)$. Then $\text{Type}(T(\xi)) = [\xi]$.

In particular we have the *trivial* bispace $T = T(1)$ whose structure map is $t: G \rightarrow H$ and for which the induced G bispace is just G with the usual left and right G action.

Example 2.4. Let X be a right A space for an abelian group A . We can make X an A bispace by defining $axb = xba^{-1}$ and with this definition X has structure map ψ defined by $\psi(x)(a) = a^{-1}$. Note that

$$A \rightarrow 1 \rightarrow \text{Aut}(A)$$

is a crossed module precisely when A is abelian. A right A space is then a $(1, A)$ bispace with structure map equal to 1. It is also possible to consider $(\text{Aut}(A), A)$ bispaces such as the Jandl bundle discussed below in Example 3.7.

If X and Y are G bispaces then a function $f: X \rightarrow Y$ is called a G bispace *morphism* if it commutes with the left and right actions. Note that, just as in the case of ordinary G -spaces, every morphism of G -bispaces is an isomorphism. We have the following lemma (see Remarque 2.7 of [4]).

Lemma 2.4. *If X and Y are G bispaces and $f: X \rightarrow Y$ is a bijection then f is a G bispace isomorphism if and only if it commutes with the right action and the structure map.*

Proof. Follows from Lemma 2.1. □

It is then natural to make the following definition.

Definition 2.5. If X and Y are (H, G) bispaces then a function $f: X \rightarrow Y$ is called an (H, G) bispace *morphism* if it commutes with the right action and the structure maps.

Denote by $\mathbf{Bisp}_{(H, G)}$ the category of all (H, G) bispaces and bispace morphisms. In the special case of the crossed module $(\text{Aut}(G), G)$ associated to a group G , we will denote the corresponding category of bispaces by \mathbf{Bisp}_G . Note that all (H, G) bispace morphisms are automatically isomorphisms and hence $\mathbf{Bisp}_{(H, G)}$ is in fact a groupoid. We have the following proposition.

Proposition 2.6. *Two (H, G) bispaces X and Y are isomorphic if and only if they have the same type.*

Proof. Denote by $\psi: X \rightarrow H$ and $\chi: Y \rightarrow H$ the structure maps of X and Y respectively. If $f: X \rightarrow Y$ is an isomorphism then clearly $\chi = \psi \circ f$ and hence the type of Y is equal to the type of X .

Conversely assume the types are both equal to $\xi \in H/t(G)$. Notice that χ and ψ are onto the preimage of ξ in H so we can choose $x \in X$ and $y \in Y$ such that $\psi(x) = \chi(y)$. Define $f: X \rightarrow Y$ by $f(xg) = yg$. Then f is a bijection and commutes with the right G action by construction. Moreover the G equivariance of the structure maps shows that $\chi \circ f = \psi$ giving the required result. □

We can interpret this result as saying that there is a functor $\text{Type}: \mathbf{Bisp}_{(H,G)} \rightarrow H/t(G)$, where $H/t(G)$ is considered as a *discrete* groupoid, i.e. there are no non-identity morphisms.

The groupoid $\mathbf{Bisp}_{(H,G)}$ has extra structure: there is a functor

$$\otimes: \mathbf{Bisp}_{(H,G)} \times \mathbf{Bisp}_{(H,G)} \rightarrow \mathbf{Bisp}_{(H,G)}$$

which sends a pair of bispaces (X, Y) to the *product* bispace $X \otimes Y$ which is defined as follows. If (X, ψ_X) and (Y, ψ_Y) are (H, G) bispaces then $X \otimes Y$ is defined to be the bispace $X \otimes Y = (X \times Y)/G$, where G acts on $X \times Y$ by

$$(x, y)g = (xg, g^{-1}y) = (xg, y(\alpha\psi_Y(y))^{-1}(g^{-1})).$$

Denote the equivalence class of (x, y) in $X \otimes Y$ by $x \otimes y$. There is a natural right action of G on $X \otimes Y$ given by $(x \otimes y)g = x \otimes (yg)$. Define a map $\psi: X \otimes Y \rightarrow H$ by $\psi(x \otimes y) = \psi_X(x)\psi_Y(y)$. It is straightforward to check that this is well-defined and a structure map for $X \otimes Y$ making $(X \otimes Y, \psi)$ an (H, G) space. The left action induced by this structure map can be calculated using equation (2.1) to be $g(x \otimes y) = gx \otimes y$.

It is also straightforward to check that the process of forming products of bispaces is functorial. Just as is the case when forming tensor products of modules, the product of bispaces is not strictly associative, however it is associative up to a canonical natural isomorphism. Note also that the type of the product bispace satisfies $\text{Type}(X \otimes Y) = \text{Type}(X) \text{Type}(Y)$.

Example 2.5. If $\xi \in H$ we denote $X \otimes T(\xi)$ by $X(\xi)$. It is straightforward to show that $[x, g] \mapsto x\alpha(\xi)(g)$ defines a bijection from $X \otimes T(\xi)$ to X with inverse $x \mapsto [x, 1]$. With this identification the type map is $x \mapsto [\psi(x)\xi]$ and the right action is $xg = x\alpha(\xi)(g)$.

There is also a functor $(-)^*: \mathbf{Bisp}_{(H,G)} \rightarrow \mathbf{Bisp}_{(H,G)}$ which sends a bispace X to its *dual* X^* . If (X, ψ) is an (H, G) bispace we define the dual (H, G) bispace (X^*, ψ^*) to be the same underlying space X , but with the structure map $\psi^* = \psi^{-1}$ and the right group action $x \cdot g = x\alpha(\psi^{-1}(x))(g^{-1})$. Again, using equation (2.1) it can be seen that $g \cdot x = g^{-1}x$.

The following lemma is straightforward.

Lemma 2.7. *For any (H, G) bispace X we have canonical isomorphisms $X \otimes T \cong T \otimes X \cong X$ and $X \otimes X^*$ isomorphic to T .*

Write $\pi_0 \mathbf{Bisp}_{(H,G)}$ for the set of isomorphism classes in the groupoid $\mathbf{Bisp}_{(H,G)}$. This is a pointed set, pointed by the isomorphism class of the trivial bispace. The functors \otimes and $(-)^*$ induce a corresponding product and notion of dual on $\pi_0 \mathbf{Bisp}_{(H,G)}$. Since the process of forming products of bispaces is associative up to a canonical natural isomorphism the product on $\pi_0 \mathbf{Bisp}_{(H,G)}$ is associative. Lemma 2.7 shows that in fact $\pi_0 \mathbf{Bisp}_{(H,G)}$ has the structure of a group.

The fact that the type map for bispaces preserves products means that the functor $\text{Type}: \mathbf{Bisp}_{(H,G)} \rightarrow H/t(G)$ induces a homomorphism of groups

$$\text{Type}: \pi_0 \mathbf{Bisp}_{(H,G)} \rightarrow H/t(G).$$

Proposition 2.6 shows that this map is injective, and Example 2.3 shows that it is surjective. Hence Type is an isomorphism of groups. We summarize this discussion in the next proposition.

Proposition 2.8. *The type map induces an isomorphism of groups*

$$\text{Type}: \pi_0 \mathbf{Bisp}_{(H,G)} \cong H/t(G).$$

2.2. Extension and reduction of bispaces. Recall that we denote the kernel of $t: G \rightarrow H$ by G_1 . We have seen above that $G_1 \subset Z(G)$. Notice that if $h \in H$ and $g \in G_1$ then $t(\alpha(h)(g)) = ht(g)h^{-1} = 1$ so that $\alpha(h)(G_1) \subset G_1$. Hence there is an action of H on $t(G) = G/G_1$ via α and a crossed module

$$t(G) \hookrightarrow H \xrightarrow{\alpha} \text{Aut}(t(G)).$$

We have the following lemma.

Lemma 2.9. *If X is an (H, G) bispace then X/G_1 , with the right G -action and structure map induced from X , is an $(H, t(G))$ bispace.*

If $G \rightarrow H$ is a crossed module we say a *crossed submodule* is a crossed module $G_0 \rightarrow H_0$ with the property that G_0 is a subgroup of G , H_0 is a subgroup of H , $t(G_0) \subset H_0$ and the elements in $\alpha(H_0) \subset \text{Aut}(G)$ fix G_0 and thus define a homomorphism $H_0 \rightarrow \text{Aut}(G_0)$. In such a case $G_0 \rightarrow H_0$ is clearly a crossed module and the inclusions define a morphism of crossed modules.

Example 2.6. If $t: G \rightarrow H$ is a crossed module and $G_1 = \ker(t)$ then $(1, G_1)$ is a crossed submodule of (H, G) where 1 is the identity subgroup of H .

Let X be an (H, G) bispace and (H_0, G_0) a crossed-submodule of (H, G) . We say that $X_0 \subset X$ is a *reduction* of X to (H_0, G_0) if X_0 is an orbit of G_0 and $\psi(X_0) \subset H_0$. Clearly X_0 is an (H_0, G_0) bispace with structure map ψ_{X_0} .

Let X_1 denote the subspace

$$(2.2) \quad X_1 = \{x \in X \mid \psi(x) = 1\}.$$

Then we have the following lemma.

Lemma 2.10. *Let X be an (H, G) bispace then*

- (1) X_1 is non-empty if and only if $\text{Type}(X) = 1$.
- (2) If X_1 is non-empty then it is a reduction of X to $(1, G_1)$.

Proof. The first part is obvious from the definition. If X_1 is non-empty and if $x \in X_1$ and $g \in G_1$ then $\psi(xg) = \psi(x)t(g) = 1$. On the other hand if $x, y \in X_1$ then $x = yg$ for some $g \in G$ and $1 = \psi(x) = \psi(y)t(g) = t(g)$ so that $g \in G_1$. \square

Example 2.7. If X is a G bispace then $t = \text{Ad}: G \rightarrow \text{Aut}(G) = H$ so that $G_1 = Z(G)$. Then X_1 is either empty or a $(1, Z(G))$ bispace.

Let (H, G) and (H', G') be crossed modules, $(\zeta, \eta): (H, G) \rightarrow (H', G')$ a morphism of crossed modules and X an (H, G) bispace. Let $X(G') = X \times_G G'$ where the action of G on $X \times G'$ is $(x, g')g = (xg, \eta(g^{-1})g')$. Clearly $X(G')$ is a right G' space with action $[x, g']g'' = [x, g'g'']$. Define a structure map by $\psi([x, g']) = \zeta(\psi(x))t'(g')$. This is well-defined and equivariant making $X(G')$ an (H', G') bispace. We call $X(G')$ the *extension* of X to (H', G') . In particular we have the following Lemma whose proof is straightforward.

Lemma 2.11. *Let X_0 be a reduction of the (H, G) bispace X to the crossed submodule (H_0, G_0) . Then the map $X_0(G) \rightarrow X$ defined by $[x, g] \mapsto xg$ defines an isomorphism of (H, G) bispaces.*

Thus we have also:

Lemma 2.12. *If $\text{Type}(X) = 1$ then $X \simeq X_1(G)$.*

Note 2.1. As additional motivation for our introduction of crossed modules we note that if we have a G bispace X and a homomorphism $G \rightarrow H$ there is no natural induced H bispace. We need the additional data of a homomorphism $\text{Aut}(G) \rightarrow \text{Aut}(H)$ so that the two homomorphisms give rise to a homomorphism of crossed modules $(\text{Aut}(G), G) \rightarrow (\text{Aut}(H), H)$. Indeed this is the key reason for considering crossed modules as coefficient objects for nonabelian gerbes - we don't have functoriality in G when considering the corresponding cohomology theory, only functoriality for maps of crossed modules.

2.3. 2-groups and crossed modules. Bispaces and crossed modules are closely related to 2-groups. Recall (see for instance [2, 4]) that a (topological) *2-group* is a groupoid object \mathcal{G} in the category of topological groups. We will not spell out what this means precisely: suffice it to say that it means that \mathcal{G} is a groupoid for which both the objects and morphisms have the structure of topological groups. Since we will only ever be interested in topological 2-groups we will omit the adjective 'topological'.

Every crossed module $t: G \rightarrow H$ uniquely determines a 2-group \mathcal{G} and vice versa. The group of objects of \mathcal{G} is defined to be H , while the group of morphisms of \mathcal{G} is defined to be the semi-direct product $H \rtimes G$. For more details we refer to [2, 4].

The notion of 2-group that we have surveyed here is a 'strict' one; there is also a notion of *weak* 2-group for which we refer to [2]. Put briefly, a weak 2-group is a (topological) 2-groupoid with one object; from another point of view a weak 2-group consists of a groupoid \mathcal{G} equipped with a functor $\otimes: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ which is associative up to a coherent natural isomorphism, together with several other axioms.

The groupoid $\mathbf{Bisp}_{(H,G)}$ of all (H, G) bispaces is a prime example of a weak 2-group in this sense. As we have seen, the product $X \otimes Y$ of two (H, G) bispaces defines a functor

$$\otimes: \mathbf{Bisp}_{(H,G)} \times \mathbf{Bisp}_{(H,G)} \rightarrow \mathbf{Bisp}_{(H,G)}$$

as above. There is a canonical functor from the groupoid $\mathbf{Bisp}_{(H,G)}$ to the groupoid \mathcal{G} associated to the crossed module (H, G) . This canonical functor preserves products in $\mathbf{Bisp}_{(H,G)}$ and \mathcal{G} in an appropriately weakened sense: it turns out that this canonical functor is an *equivalence* between the weak 2-group $\mathbf{Bisp}_{(H,G)}$ and its strict version \mathcal{G} . For more details we refer to [2].

This last statement is partially analogous to the following well known fact about topological groups: if G is a topological group and \mathbf{Sp}_G denotes the groupoid of right G spaces and maps between them, then \mathbf{Sp}_G is equivalent to G , thought of as a groupoid with one object. The difference between this situation and the one we are considering lies in the fact that the groupoid G is not normally a 2-group, in fact it is a 2-group if and only if G is abelian.

The groups $\ker(t)$ and $H/t(G)$ also have a nice interpretation in terms of 2-groups. It turns out (using the technology of simplicial homotopy theory, see for instance [10]) that one can make sense of the homotopy groups $\pi_i(\mathcal{G})$ of a 2-group \mathcal{G} . In fact the crossed module $G \rightarrow H$ associated to a 2-group \mathcal{G} arises in this setting as the *Moore complex* of the simplicial group which is the nerve of \mathcal{G} . If

one follows the standard recipe for computing the simplicial homotopy groups of a simplicial group then one finds that the homotopy groups $\pi_i(\mathcal{G})$ correspond to the homology groups of $G \rightarrow H$, thought of as a complex concentrated in degrees 0 and 1. So one finds that $\pi_0(\mathcal{G}) = H/t(G)$ and that $\pi_1(\mathcal{G}) = \ker(t)$. The fact that $t(G)$ is normal in H and the fact that $\ker(t)$ is abelian can then be understood as higher dimensional analogues of the fact that the set of path components of a topological group has the structure of a group and the fact that the fundamental group of a topological group is abelian, respectively.

3. BIBUNDLES

Definition 3.1. Let (H, G) be a crossed module. If $P \rightarrow M$ is a (right) principal G bundle with an equivariant map $\psi: P \rightarrow H$ such that each fibre of $P \rightarrow M$ is a (H, G) bispace we call $P \rightarrow M$ an (H, G) *bibundle*.

We will call P the *total space* and M the *base space* of an (H, G) bibundle $P \rightarrow M$. As for the case of bispaces we say that a *morphism* of bibundles is a morphism of the underlying principal bundles which commutes with the structure maps. Clearly every morphism of bibundles inducing the identity on base spaces is an isomorphism. We will write $\mathbf{Bibun}_{(H, G)}(M)$ for the groupoid of bibundles on M and we will denote the set of isomorphism classes of bibundles on M by $\pi_0 \mathbf{Bibun}_{(H, G)}(M)$. If P is a bibundle on M then we will write $[P]$ for its isomorphism class in $\pi_0 \mathbf{Bibun}_{(H, G)}(M)$.

Consider an (H, G) bibundle $P \rightarrow M$. Each fibre of $P \rightarrow M$ is an (H, G) bispace so it follows immediately from the discussions in Section 2 that we have a commuting diagram

$$(3.1) \quad \begin{array}{ccc} P & \xrightarrow{\psi} & H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & H/t(G) \end{array}$$

where ψ satisfies $pg = (\alpha \circ \psi)(p)(g)p$. As before we will call ψ the *structure map* of P and ϕ the *type map* of P . Local triviality of $P \rightarrow M$ will ensure that ψ and ϕ are smooth or continuous as appropriate.

Note 3.1. We remark that the notion of (H, G) bibundle (and the notion of (H, G) bispace) is actually a special case of the notion of groupoid bundle. Recall that if \mathcal{G} is a topological groupoid with space of objects G_0 and space of morphisms G_1 , then a \mathcal{G} *groupoid bundle* over M (see [7, 11]) consists of a map $\pi: P \rightarrow M$ admitting local sections together with an *action* of \mathcal{G} on P , in other words the data of

- (1) a map $p: P \rightarrow G_0$,
- (2) a map $m: P \times_{G_0} G_1 \rightarrow P$

satisfying certain axioms (for which refer to [7, 11]). Furthermore the action is required to be principal in the sense that the diagram

$$\begin{array}{ccc} P \times_{G_0} G_1 & \xrightarrow{m} & P \\ p_1 \downarrow & & \downarrow \\ P & \longrightarrow & M \end{array}$$

is a pullback, where p_1 denotes projection onto the first factor. When \mathcal{G} is the groupoid associated to a crossed module (H, G) as described in Subsection 2.3 above, the notion of \mathcal{G} -bundle coincides exactly with the notion of (H, G) -bibundle.

Example 3.1. If $P \rightarrow M$ is a (right) principal A bundle for an abelian group A then $P \rightarrow M$ is a $(1, A)$ bibundle with structure map $1: P \rightarrow 1$.

Example 3.2. Suppose that G is a normal subgroup of H so that we have a crossed module $i: G \rightarrow H$. Let K denote the quotient group H/G and suppose that the projection $H \rightarrow K$ admits local sections, then $H \rightarrow K$ is an (H, G) bibundle. In this case the structure map $\psi: H \rightarrow H$ is the identity.

Example 3.3. Similarly if (H, G) is a crossed module then $H \rightarrow H/t(G)$ is an $(H, t(G))$ bibundle with structure map $\psi: H \rightarrow H$ equal to the identity. Notice that the fibre H_ξ over $\xi \in H/t(G)$ is a $t(G)$ bispace of type ξ .

Example 3.4. As an important example of the above construction let K be a simple, simply-connected, compact Lie group and denote by PK the group of all smooth maps $k: [0, 1] \rightarrow K$ with $k(0) = 1$. If we define $\pi: PK \rightarrow K$ to be evaluation of a path at 1 then this is an ΩK bibundle. Here we are defining the loop group $\Omega K \subset PK$ to be the subgroup of all paths k with $k(0) = k(1)$. Note that this is a larger group than the group of smooth maps k from S^1 to K with $k(0) = 1$. As in the general case above the adjoint action of PK on itself fixes the subgroup ΩK so we have a crossed module

$$\Omega K \rightarrow PK \rightarrow \text{Aut}(\Omega K)$$

and thus the $(PK, \Omega K)$ bibundle $PK \rightarrow K$.

Example 3.5. If $P \rightarrow M$ is an (H, G) bibundle and G_1 is the kernel of $t: G \rightarrow H$ then $P/G_1 \rightarrow M$ is a $(H, t(G))$ bibundle, where $t(G) = G/G_1$.

Just as with the structure and type maps many of the other notions we have introduced for bispaces can be extended immediately to bibundles by applying them to the fibres of $P \rightarrow M$. In particular this applies to the notions of reduction and extension and the product and dual constructions. So if $P \rightarrow M$ and $Q \rightarrow M$ are (H, G) bibundles then there are bibundles $P^* \rightarrow M$ and $P \otimes Q \rightarrow M$. If P, Q and R are bibundles on M then there are canonical isomorphisms $P \otimes (Q \otimes R) \cong (P \otimes Q) \otimes R$ and $P \otimes P^* \cong T$ where $T = M \times G$ is the trivial bibundle on M whose fibre at each point of M is the trivial bispace. In a completely analogous way to the earlier discussion for bispaces, we have the following Lemma.

Lemma 3.2. *The set $\pi_0 \mathbf{Bibun}_{(H, G)}(M)$ of isomorphism classes of bibundles on M forms a group with product $[P] \otimes [Q]$ defined by $[P \otimes Q]$ and where the inverse $[P]^{-1}$ of an element $[P]$ is given by $[P^*]$.*

Example 3.6. If $H \rightarrow K$ is the bibundle of Example 3.2 above then the product $H \times H \rightarrow H$ in H induces a bibundle morphism $H \otimes H \rightarrow H$ covering the product in K so that the diagram

$$\begin{array}{ccc} H \otimes H & \longrightarrow & H \\ \downarrow & & \downarrow \\ K \times K & \longrightarrow & K \end{array}$$

commutes. Similarly the inverse map $h \mapsto h^{-1}$ in H defines an isomorphism $H \rightarrow H^*$ covering the inverse map in K .

Note that (3.1) is not quite a morphism of (H, G) bibundles, instead

$$\begin{array}{ccc} P/G_1 & \xrightarrow{\psi} & H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & H/t(G) \end{array}$$

is a morphism of $(t(G), H)$ bibundles. From this discussion we deduce the following proposition.

Proposition 3.3. *If $P \rightarrow M$ is an (H, G) bibundle with type map $\phi: M \rightarrow H/t(G)$ then P/G_1 is the pull-back of $H \rightarrow H/t(G)$ by ϕ .*

As a consequence we can deduce the following corollary.

Corollary 3.4. *If $t: G \rightarrow H$ is a crossed module with $\ker(t) = 1$ then $G \simeq t(G)$ and every bibundle $P \rightarrow M$ is the pullback of the (H, G) bibundle $H \rightarrow H/t(G)$ by the type map.*

Consider the case when the type map is the constant map to the identity in $H/t(G)$. Then each fibre of $P \rightarrow M$ has a non-empty subset of points $p \in P$ such that $\psi(p) = 1$. By analogy with (2.2) above Lemma 2.10 denote the union of these subsets by P_1 and note that from Lemma 2.10 we have that P_1 is a reduction of P to $(1, t(G))$.

Following [1] we say that a section s of P is a *central* section if $\psi \circ s = 1$. We then have the following proposition.

Proposition 3.5 (c.f [1]). *A bibundle P is trivial if and only if it has a central section.*

This gives us immediately the following result.

Corollary 3.6. *A bibundle P is trivial if and only if the type map is equal to 1 and P_1 is trivial.*

If $Q \rightarrow M$ is a G_1 bundle, that is a $(1, G_1)$ bibundle, then we can extend to an (H, G) bibundle $\iota(Q) = Q(G)$ using the construction from Subsection 2.2. So we have $\iota(Q) = Q \times_{G_1} G$ and $\psi([q, g]) = t(g) \in H$. The type of $\iota(Q)$ is clearly $1 \in \text{Map}(M, H/t(G))$. On the other hand assume that P has type 1 so that we have a well-defined reduction of P_1 to $(1, G_1)$. Then from Lemma 2.11 we have the isomorphism

$$\begin{array}{ccc} P_1 \times_{G_1} G & \simeq & P \\ [p, g] & \mapsto & pg. \end{array}$$

Hence we have the following proposition.

Proposition 3.7. *$Q \rightarrow M$ is a trivial G_1 bundle if and only if $\iota(Q)$ is a trivial (H, G) bibundle.*

Proof. Clearly if Q has a section then it induces a section of $\iota(Q)_1$ which is a central section on $\iota(Q)$ so by Corollary 3.6 Q is trivial. On the other hand if $\iota(Q)$ is trivial then it has a central section. But that must be a section of $\iota(Q)_1 \simeq Q$, thus Q has a section and is trivial. \square

It follows that we have a sequence of 2-groups and homomorphisms between them

$$(3.2) \quad 1 \rightarrow \mathbf{Bun}_{G_1}(M) \xrightarrow{\iota} \mathbf{Bibun}_{H,G}(M) \xrightarrow{\text{Type}} \text{Map}(M, H/t(G)),$$

where the group $\text{Map}(M, H/t(G))$ is thought of as a discrete 2-group. This sequence is ‘exact’ in the following sense. The homomorphism ι is faithful, and if P is a bibundle on M then $\text{Type}(P) = 1$ if and only if P is isomorphic to a bibundle of the form $\iota(R)$, where R is a G_1 bundle on M .

On passing to isomorphism classes we obtain the exact sequence of sets

$$(3.3) \quad 1 \rightarrow \pi_0 \mathbf{Bun}_{G_1}(M) \rightarrow \pi_0 \mathbf{Bibun}_{H,G}(M) \xrightarrow{\text{Type}} \text{Map}(M, H/t(G)).$$

Consider now the image of the type map. To understand this let $\phi: M \rightarrow H/t(G)$ be any map. We can pullback the $t(G)$ bundle $H \rightarrow H/t(G)$ along this map so that we get a pullback diagram

$$\begin{array}{ccc} \phi^*(H) & \xrightarrow{\psi} & H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & H/t(G). \end{array}$$

Assume we can find a G bundle $P \rightarrow M$ which lifts the (right) $t(G)$ bundle $\phi^*(H) \rightarrow M$ to a (right) G bundle. Then we can define a map $\psi: P \rightarrow H$ by the composite $P \rightarrow \phi^*(H) \rightarrow H$. This map ψ is equivariant and hence is the structure map for an (H, G) bibundle structure on $P \rightarrow M$ with ϕ as type map. We conclude that for ϕ to be in the image of Type it suffices for us to be able to lift $\phi^*(H)$ to a G bundle. Consider then the central extension

$$0 \rightarrow G_1 \rightarrow G \rightarrow t(G) \rightarrow 0.$$

The obstruction to lifting $\phi^*(H)$ from $t(G)$ to G is the non-triviality of the class in $H^2(M, G_1)$ of the G_1 lifting bundle gerbe [12] associated to $\phi^*(H)$. It follows that we have the exact sequence of groups

$$(3.4) \quad 1 \rightarrow \pi_0 \mathbf{Bun}_{G_1}(M) \xrightarrow{\iota} \pi_0 \mathbf{Bibun}_{H,G}(M) \xrightarrow{\text{Type}} \text{Map}(M, H/t(G)) \xrightarrow{\epsilon} H^2(M, G_1).$$

We remark that there is an alternative way to arrive at this exact sequence, for which we sketch the details. The 2-category $\mathbf{BGrb}_{G_1}(M)$ of G_1 bundle gerbes on M together with the corresponding 1-morphisms and 2-morphisms between them, is an example of a 3-group. The map which sends a map $\phi: M \rightarrow H/t(G)$ to the lifting G_1 bundle gerbe on M determined by the pullback G_1 bundle ϕ^*H defines a homomorphism $\text{Map}(M, H/t(G)) \rightarrow \mathbf{BGrb}_{G_1}(M)$. We can extend the exact sequence (3.2) one term to the right and obtain an exact sequence of 3-groups

$$1 \rightarrow \mathbf{Bun}_{G_1}(M) \xrightarrow{\iota} \mathbf{Bibun}_{H,G}(M) \xrightarrow{\text{Type}} \text{Map}(M, H/t(G)) \rightarrow \mathbf{BGrb}_{G_1}(M)$$

where ‘exact’ is to be understood in a similar sense to that above. Taking π_0 recovers the exact sequence (3.4) above.

The exact sequence (3.3) tells us loosely that if $H/t(G)$ is small then most (G, H) bibundles are likely to be abelian, i.e. reduce to abelian $(1, G_1)$ bibundles or G_1 bundles. Recall that there is a map $H/t(G) \rightarrow \text{Aut}(G)/\text{Ad}(G) = \text{Out}(G)$ so the question of whether or not there are many (G, H) bibundles that are not abelian also relates to the size of $\text{Out}(G)$ which we consider next.

3.1. Type maps that lift. We say that a map $\phi: M \rightarrow H/t(G)$ *lifts* if there is some $\hat{\phi}: M \rightarrow H$ which projects to ϕ . The construction used in Example 2.3 can be applied to define a bibundle $T(\hat{\phi})$ whose fibre over $x \in M$ is $T(\hat{\phi}(x))$. This gives a map

$$\text{Map}(M, H) \rightarrow \pi_0 \mathbf{Bibun}_{(H,G)}(M)$$

which makes the following diagram commute

$$(3.5) \quad \begin{array}{ccc} \text{Map}(M, H) & \xrightarrow{\quad} & \pi_0 \mathbf{Bibun}_{(H,G)}(M) \\ & \searrow & \downarrow \text{Type} \\ & & \text{Map}(M, H/t(G)). \end{array}$$

Recall also from Example 2.5 that if $Q \rightarrow M$ is a bibundle we denote $Q(\hat{\phi}) = Q \otimes T(\hat{\phi})$.

As a result we have the following proposition.

Proposition 3.8. *If $P \rightarrow M$ is a bibundle with type map $\phi: M \rightarrow H/t(G)$ which lifts to $\hat{\phi}: M \rightarrow H$ then P is isomorphic to $\iota(R)(\hat{\phi})$ for some G_1 bundle R .*

Proof. By construction $T(\hat{\phi})$ has type map ϕ , the same as P . Hence there is a G_1 bundle R with $P \simeq \iota(R)(\hat{\phi})$. \square

In particular we deduce the following when G is a simply-connected and semi-simple Lie group.

Proposition 3.9. *If G is a simply-connected and semi-simple Lie group then every G bibundle is of the form $\iota(R)(\hat{\phi})$ for some $\hat{\phi}: M \rightarrow \text{Aut}(G)$.*

Proof. We have that $H/t(G) = \text{Aut}(G)/\text{Ad}(G) = \text{Out}(G)$ is the group of automorphisms of the Dynkin diagram of G and hence discrete. It follows that the type map ϕ is constant on connected components of M and hence lifts. \square

Example 3.7. Consider an $(\text{Aut}(A), A)$ bibundle where A is abelian. In this case $\text{Ad}(A) = 1$ so that ϕ lifts and the lift is, in fact, just $\hat{\phi} = \phi$. Hence from Proposition 3.9 every bibundle has the form $P(\hat{\phi})$ for some A -bundle P . A particular case is $A = U(1)$ when $\text{Aut}(U(1)) = \mathbb{Z}_2$. In this case we have a $U(1)$ bundle $P \rightarrow M$ and on each connected component we give it a left action by defining $zp = pz^{\pm 1}$ depending on whether ϕ restricted to that connected component is ± 1 . A $(\mathbb{Z}_2, U(1))$ bibundle is a *Jandl bundle* [14].

Note 3.2. Just as principal G -bundles on M can be described in terms of local data via their transition cocycles $g_{ij}: U_i \cap U_j \rightarrow G$ relative to some open cover $\{U_i\}$ of M , so also do bibundles have such a local description. If P is an (H, G) bibundle on M for some crossed module (H, G) then, for a sufficiently fine open cover $\{U_i\}$ of M , we can associate to P families of maps $g_{ij}: U_i \cap U_j \rightarrow G$ and $h_i: U_i \rightarrow H$ satisfying the cocycle conditions

$$\begin{aligned} g_{ij}g_{jk} &= g_{ik} \\ h_j &= h_i t(g_{ij}). \end{aligned}$$

The maps g_{ij} are the usual transition cocycles of the bundle P and arise from comparing trivializations of P on overlapping patches. The maps h_i are formed

by composing the section defining a trivialization of P over U_i with the structure map $P \rightarrow H$. One can introduce an equivalence relation on pairs (g_{ij}, h_i) and form a cohomology group $H^0(M, \mathcal{G})$ which parametrizes isomorphism classes of (H, G) bibundles.

3.2. The bundle of bibundle structures on a principal bundle. Note that the structure map ψ of an (H, G) bibundle can be viewed as a section of the associated bundle $P \times_G H$ where G acts on the right of both P and H , the latter through the action $h \cdot g = ht(g)$.

Another way of thinking about this is that if P is a right G bundle then each element in $P \times_G H$ over $x \in M$ defines a bispace structure on the right G space P_x . This is because each such element defines a structure map $P_x \rightarrow H$: if $[p, h] \in P \times_G H$ then define a map $P_x \rightarrow H$ which sends $pg \mapsto ht(g)$. Clearly this is well defined and by construction it is equivariant.

Note that bibundles pullback along maps: if P is an (H, G) bibundle on M and $f: N \rightarrow M$ is a map, then $f^*P = N \times_M P$ has a natural structure of an (H, G) bibundle on N . Therefore, if we let $\pi: P \times_G H \rightarrow M$ be the projection map, then $\pi^*(P) \rightarrow P \times_G H$ has a natural structure of an (H, G) bibundle. We can identify $P \times H$ with $\pi^*(P) \subset P \times P \times_G H$ by the map $(p, h) \mapsto (p, [p, h])$ and hence induce an (H, G) bibundle structure on $P \times H \rightarrow P \times_G H$. This is given by

$$(p, h)g = (pg, ht(g))$$

and the structure map is the projection to H . Thus we have a commutative diagram

$$\begin{array}{ccc} P \times H & \xrightarrow{\Psi} & H \\ \downarrow & & \downarrow \\ P \times_G H & \xrightarrow{\Phi} & H/t(G). \end{array}$$

In summary then we see that a (H, G) bibundle structure on P is a section $M \rightarrow P \times_G H$ and the pullback via this section of $P \times H \rightarrow P \times_G H$ is naturally isomorphic to $P \rightarrow M$ as an (H, G) bibundle.

3.3. The universal case. Let G be a topological group and let $EG \rightarrow BG$ be the universal G bundle. We can apply the construction of Section 3.2 to the right G space EG and form the space

$$EG \times H.$$

As we saw this is a bibundle over $EG \times_G H$ with right G action given by

$$(3.6) \quad (e, h)g = (eg, ht(g))$$

and structure map the projection onto H . It follows that the type map $\phi: EG \times_G H \rightarrow H/t(G)$ sends $[e, h]$ to the equivalence class of h in $H/t(G)$ and we have the commuting diagram

$$(3.7) \quad \begin{array}{ccc} EG \times H & \xrightarrow{\Psi} & H \\ \downarrow & & \downarrow \\ EG \times_G H & \xrightarrow{\Phi} & H/t(G). \end{array}$$

If $P \rightarrow M$ is an (H, G) bibundle then there is a classifying map $f: M \rightarrow BG$ which lifts to a right G equivariant map $\hat{f}: P \rightarrow EG$. Together with the structure map $\psi: P \rightarrow H$ this defines a homomorphism of (H, G) bibundles

$$\begin{aligned} \hat{\Gamma} : P &\rightarrow EG \times H \\ p &\mapsto (\hat{f}(p), \psi(p)). \end{aligned}$$

Denote by $\Gamma: M \rightarrow EG \times_G H$ the induced map. Then we have a commuting diagram of bibundles

$$(3.8) \quad \begin{array}{ccccc} P & \xrightarrow{\hat{\Gamma}} & EG \times H & \xrightarrow{\Psi} & H \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{\Gamma} & EG \times_G H & \xrightarrow{\Phi} & H/t(G). \end{array}$$

As a concrete example of this we have the following example.

Example 3.8. Consider the crossed module $\Omega K \rightarrow PK$ so that $G = \Omega K$ and $H = PK$. We can realise $EG \rightarrow BG$ as the path fibration $PK \rightarrow K$. Then $EG \times H = PK \times PK$ with the right action of ΩK . The diagram above becomes

$$(3.9) \quad \begin{array}{ccccc} P & \xrightarrow{\hat{\Gamma}} & PK \times PK & \xrightarrow{\Psi} & PK \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{\Gamma} & PK \times_{\Omega K} PK & \xrightarrow{\Phi} & K. \end{array}$$

The function Γ in (3.8) is a candidate for a classifying map for bibundles. However, before we can justify this we need to study how bibundles induced by pullback are related to one another under homotopies of maps.

3.4. Homotopy of bibundles. A key fact in the theory of ordinary bundles is the fact that homotopic maps induce isomorphic bundles under pullback. This same statement fails to be true for bibundles. To see why, recall that the type map $M \rightarrow H/t(G)$ of an (H, G) bibundle P on M is an invariant of the isomorphism class of P . In other words, if P and Q are isomorphic bibundles on M then the type map of P is *equal* to the type map of Q . If $h: M \times I \rightarrow N$ is a homotopy between maps $f_0, f_1: M \rightarrow N$ and P is a bibundle on N then there is no reason why the type map of the induced bundle h^*P should be constant in the t -direction (here t is the coordinate on the interval I), and hence there is no reason why the bibundles f_0^*P and f_1^*P should be isomorphic.

To get around this problem we clearly need to restrict our attention to homotopies $h: M \times I \rightarrow N$ which satisfy the following property: the composite map $\phi \circ h: M \times I \rightarrow H/t(G)$ is constant in the t -direction, where ϕ is the type map of P . Put another way, we have a commutative diagram

$$\begin{array}{ccc} M \times I & \xrightarrow{h} & N \\ & \searrow & \swarrow \\ & H/t(G) & \end{array} \quad \phi$$

where the map $M \times I \rightarrow H/t(G)$ is the composition $M \times I \xrightarrow{\text{pr}_1} M \xrightarrow{\phi_0} H/t(G)$, where ϕ_0 denotes the type map of h_0^*P .

Therefore, we should regard $h: M \times I \rightarrow N$ as a map in the category of spaces over $H/t(G)$, which places us in the realm of parametrized homotopy theory.

Recall that if B is a space then the category of spaces over B is the category whose objects are spaces X equipped with a map $X \rightarrow B$ (we will refer to such spaces as spaces *over* B) and whose morphisms are maps of spaces which are compatible with the projections to B (we will also refer to such maps as maps *over* B). Note that the identity map $1_B: B \rightarrow B$ exhibits B as an object of this category (this is the terminal object of the category of spaces over B). Note also that product in the category of spaces over B of two spaces X, Y over B is the fibre product $X \times_B Y$.

A homotopy between two maps $f_0, f_1: M \rightarrow N$ in the category of spaces over B is then the usual sort of map $h: M \times I \rightarrow N$, but where $h_t: M \rightarrow N$ defined by $h_t(x) = h(x, t)$ is a map over B for all $t \in I$. We will sometimes say that h is a homotopy from f_0 to f_1 *over* B . Clearly if f_0 and f_1 are homotopic maps over B , then f_0 and f_1 are homotopic in the usual sense.

In particular we have the notion of a homotopy equivalence over B , also called a *fibre homotopy equivalence*. Notice that if $f: X \rightarrow Y$ is a fibre homotopy equivalence, then for any space Z over B , the induced map $Z \times_B X \rightarrow Z \times_B Y$ is also a fibre homotopy equivalence (this fails to be true if f is just an ordinary homotopy equivalence). In particular if X is contractible as a space over B , in other words if the given map $X \rightarrow B$ is a homotopy equivalence over B , then for any space Z over B the induced map $Z \times_B X \rightarrow Z$ is a fibre homotopy equivalence.

Homotopy of maps over B is an equivalence relation and we will write $[X, Y]_B$ for the set of homotopy classes of maps over B between two spaces X and Y over B .

With this understanding of the notion of homotopy, we have the following proposition.

Proposition 3.10. *Suppose that $f_0: M \rightarrow N$ and $f_1: M \rightarrow N$ are homotopic maps in the category of spaces over $H/t(G)$, where M is paracompact. If P is a fibundle on N then the fibundles $P_0 = f_0^*P$ and $P_1 = f_1^*P$ on M are isomorphic.*

Proof. Let $h: M \times I \rightarrow N$ be a homotopy from f_0 to f_1 over $H/t(G)$ and let $Q \rightarrow M \times I$ denote the induced bundle h^*P . It suffices to show that the fibundles Q and $P_0 \times I$ on $M \times I$ are isomorphic. Standard bundle theory shows that the underlying G bundles P and $P_0 \times I$ are isomorphic.

The G bundle $P_0 \times I$ is equipped with the structure map $P_0 \times I \rightarrow P_0 \rightarrow H$ which is constant in the I -direction (the map $P_0 \rightarrow H$ is the restriction of the structure map $Q \rightarrow H$ of Q to $Q|_{M \times \{0\}}$). Using the isomorphism $Q \cong P_0 \times I$ we can define on $P_0 \times I$ a new structure map $P_0 \times I \rightarrow H$. We would like to show that these two structure maps define isomorphic fibundle structures on $P_0 \times I$.

Let $\hat{\phi}_1: P_0 \times I \rightarrow H$ denote the structure map which is constant in the I -direction and let $\hat{\phi}_2: P_0 \times I \rightarrow H$ denote the structure map which is induced by the isomorphism of G -bundles $Q \cong P_0 \times I$. Since $\hat{\phi}_1$ and $\hat{\phi}_2$ correspond to the same (constant) type map $M \times I \rightarrow H/t(G)$ we must have that $\hat{\phi}_2 = \hat{\phi}_1 \chi$ for some map $\chi: P_0 \times I \rightarrow G/\ker(t)$ which satisfies

$$R_g^* \chi = t(g)^{-1} \chi t(g).$$

Suppose that we can find a map $\hat{\chi}: P_0 \times I \rightarrow G$ satisfying $\chi = t(\hat{\chi})$ and $R_g^* \hat{\chi} = g^{-1} \hat{\chi} g$. Then $\hat{\chi}$ defines a G bundle automorphism of $P_0 \times I$. Furthermore this bundle automorphism is compatible with $\hat{\phi}_1$ and $\hat{\phi}_2$ in the obvious sense. It follows that $\hat{\phi}_1$ and $\hat{\phi}_2$ define isomorphic bibundle structures on $P_0 \times I$.

It remains to prove the existence of the equivariant map $\hat{\chi}: P_0 \times I \rightarrow G$. We are given the equivariant map $\chi: P_0 \times I \rightarrow G/\ker(t)$ and we note that this restricts to the constant map 1 on $P_0 \times \{0\}$. Since χ is an equivariant map it can be regarded as a section s of the associated bundle of groups $P_0(G/\ker(t)) \times I$ on $M \times I$. The condition that χ restricts to the constant map on $P_0 \times \{0\}$ translates into the condition that the section s is identically 1 on $M \times \{0\}$.

The homomorphism $t: G \rightarrow G/\ker(t)$ induces a map $P_0(G) \rightarrow P_0(G/\ker(t))$ of the associated bundles. As part of our assumptions on the crossed module (H, G) (see Section 2) we assume that the map $G \rightarrow G/\ker(t)$ has local sections, and is hence locally trivial with fibre $\ker(t)$. It follows that $P_0(G) \rightarrow P_0(G/\ker(t))$ is a locally trivial fibre bundle with fibre $P_0(\ker(t))$.

We need to prove that the section s of $P_0(G/\ker(t)) \times I$ lifts to a section \hat{s} of $P_0(G) \times I$. Such a section \hat{s} can be thought of as a section of the bundle R on $M \times I$ obtained by pulling back the bundle $P_0(G) \times I \rightarrow P_0(G/\ker(t)) \times I$ with the section $s: M \times I \rightarrow P_0(G/\ker(t)) \times I$.

We are in the following situation: we have a fibre bundle $R \rightarrow M \times I$ together with a section defined over $M \times \{0\}$ and we want to extend this section to a section defined over the whole of $M \times I$. Since $R \rightarrow M \times I$ is a fibration, it has the homotopy lifting property, hence such a section exists. \square

3.5. Classifying theory for bibundles. We can consider the total space P of an (H, G) bibundle on M as an object in the category of spaces over H via the structure map. Let us say that P is *contractible* as a space over H if the structure map $P \rightarrow H$ is a homotopy equivalence in the category of spaces over H . As an example, since EG is contractible as an ordinary space, $EG \times H$ is contractible when viewed as a space over H with the map to H being projection to the second factor. We have the following analogue of the classical bundle classification theorem.

Theorem 3.11. *Suppose that E is an (H, G) bibundle over a space B such that E is contractible when viewed as a space over H . Then $E \rightarrow B$ is a universal bibundle in the sense that there is an isomorphism*

$$[M, B]_{H/t(G)} \cong \pi_0 \mathbf{Bibun}_{(H, G)}(M)$$

induced by pullback of bibundles, for any paracompact space M .

Our proof will be an adaptation of the proof of Theorem 7.5 from [6]. In this paper Dold introduces some key notions which we will recall here, as they play an important role in what follows. Recall (see Definition 2.2 of [6]) that a map $p: Y \rightarrow X$ is said to have the *section extension property* if the following is true: if A is a closed subspace of X and $s: A \rightarrow Y$ is a section of p defined over A which has an extension to a ‘halo’ around A , then there is an extension of s to a section of p defined on X . Here a *halo* of A is a subset V of X such that there is a continuous map $\tau: X \rightarrow I$ with the property that $\tau(a) = 1$ for all $a \in A$ and $\tau(x) = 0$ for all $x \in X - V$. It follows that any map $p: Y \rightarrow X$ with the section extension property admits at least one section.

Dold proves the important Theorem 2.7 of [6] which says that if $\{U_i\}$ is a numerable open cover of X and the restriction $p|_{U_i}$ of p to U_i has the section extension property for all $i \in I$, then $p: Y \rightarrow X$ has the section extension property. A sufficient condition for a map $q: U \rightarrow V$ to have the section extension property is that q is *shrinkable*, in other words q is fibre homotopy equivalent to the identity map 1_V (see Proposition 2.3 of [6]). Therefore, if there exists a numerable open cover $\{U_i\}$ of X such that $p|_{U_i}$ is shrinkable for all $i \in I$, then $p: Y \rightarrow X$ has the section extension property.

Proof. Proposition 3.10 shows that the map $[M, B]_{H/t(G)} \rightarrow \pi_0 \mathbf{Bibun}_{(H,G)}(M)$ which sends a homotopy class $[f]$ of maps over $H/t(G)$ to the isomorphism class of the pullback bibundle f^*E is well defined. We want to show that this map is an isomorphism. We first show that this map is surjective. Let P be a bibundle on M and consider the space $P \times_H E$ over M and its quotient $(P \times_H E)/G$ by the diagonal G -action. A section of the canonical map $(P \times_H E)/G \rightarrow M$ is a G -equivariant map $P \rightarrow E$ which is compatible with the structure maps of P and E . If such a section exists, then we have a map $f: M \rightarrow B$ and it follows that P is isomorphic to the pullback f^*E .

First suppose that (P, ϕ) is a bibundle on M such that the underlying principal G bundle is trivial — suppose that $s: M \rightarrow P$ is a section of the map $\pi: P \rightarrow M$. We will show that in this case the map $(P \times_H E)/G \rightarrow M$ is shrinkable. We regard M as a space over H via the map $\phi \circ s: M \rightarrow H$. Then there is an isomorphism

$$\begin{aligned} (P \times_H E)/G &\rightarrow M \times_H E \\ [p, e] &\mapsto (\pi(p), e\tau(p, s\pi(p))) \end{aligned}$$

which is compatible with the projections to M . Here $\tau: P^{[2]} \rightarrow G$ denotes the usual map which satisfies $p_2 = p_1\tau(p_1, p_2)$ for $(p_1, p_2) \in P^{[2]}$. It is easy to see that this map is well defined. To see that $(\pi(p), e\tau(p, s\pi(p))) \in M \times_H E$ as claimed, note that if $(p, e) \in P \times_H E$ is a representative of $[p, e]$ then $\phi(p) = \psi(e)$, where $\psi: E \rightarrow H$ is the structure map of the bibundle E . Therefore

$$\psi(e\tau(p, s\pi(p))) = \phi(p)\tau(p, s\pi(p)) = \phi(p\tau(p, s\pi(p))) = \phi(s\pi(p)),$$

as required. An inverse for the map above is given by the map $M \times_H E \rightarrow (P \times_H E)/G$ which sends $(m, e) \mapsto [s(m), e]$ as is easily checked. Since E is contractible as a space over H , we see that $M \times_H E \rightarrow M$, and hence $((M \times G) \times_H E)/G \rightarrow M$, is shrinkable.

Since M is paracompact, we can form a numerable cover $\{U_i\}_{i \in I}$ of M by open sets U_i with the property that there exist local sections $s_i: U_i \rightarrow P$ of $\pi: P \rightarrow M$ over U_i . It follows from the previous argument that the restriction of $(P \times_H E)/G \rightarrow M$ to U_i is shrinkable for all $i \in I$. Hence, by Theorem 2.7 of [6], we see that $(P \times_H E)/G \rightarrow M$ admits a section over M . Hence P is induced by pullback from E via a map $M \rightarrow B$.

Next we show that the map is injective. Suppose that $f_0: M \rightarrow B$ and $f_1: M \rightarrow B$ are representatives of homotopy classes of maps for which there is a bibundle isomorphism $f_0^*E \cong f_1^*E$. Let $P_0 = f_0^*E$, $P_1 = f_1^*E$ and suppose that $\alpha: P_0 \rightarrow P_1$ is a bibundle isomorphism. Let $P = P_0 \times I$ and consider, as in [6], the bibundle map

$$(3.10) \quad P|_{M \times \{0,1\}} \rightarrow E$$

induced by α and the bibundle map $\alpha_0: P_0 \rightarrow E$. This bibundle map is a section of $(P \times_H E)/G$ over $M \times \{0, 1\}$. By the same argument as above, since E is contractible as a space over H , the map $(P \times_H E)/G \rightarrow M \times I$ is locally shrinkable and hence has the section extension property.

The section of $(P \times_H E)/G \rightarrow M \times I$ over $M \times \{0, 1\}$ defined by (3.10) has an extension to a halo around $M \times \{0, 1\}$ since it can be extended (following [6]) to a bundle map

$$\begin{aligned} P|_{M \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1])} &\rightarrow E \\ (u, t) &\mapsto \begin{cases} \alpha_0(u) & \text{if } t < \frac{1}{2} \\ \alpha_1 \alpha(u) & \text{if } t > \frac{1}{2} \end{cases} \end{aligned}$$

where we have written $\alpha_1: P_1 \rightarrow E$ for the bundle map covering map $f_1: M \times I \rightarrow B$. The set $V = M \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1])$ is a halo around $M \times \{0, 1\}$ as explained in [6]. Therefore, by the section extension property, there is an extension of the section (3.10) to a global section defined over the whole space $M \times I$. This global section corresponds to a bundle map $P_0 \times I \rightarrow E$ which covers a map $M \times I \rightarrow B$. The latter map is a homotopy between f_0 and f_1 . \square

Note 3.3. Note that this theorem need not be true if we replace homotopy classes over $H/t(G)$ with arbitrary homotopy classes. As we remarked in Subsection 3.4 above, if f_0 and f_1 are homotopic maps from M into B which are not homotopic over $H/t(G)$ then there is no reason why the structure maps of the induced bundles f_0^*E and f_1^*E should be equal, and hence no reason why f_0^*E and f_1^*E should be isomorphic bibundles. Arbitrary homotopy classes leads to the notion of ‘concordance’ of bibundles: two bibundles P_0 and P_1 on M are said to be *concordant* if there is a bibundle P on $M \times I$ such that $P|_{M \times \{0\}} \cong P_0$ and $P|_{M \times \{1\}} \cong P_1$. The concordance relation is an equivalence relation and it can be shown that the set of concordance classes of bibundles on M is in a bijective correspondence with homotopy classes of maps from M into B , if B is a space as in Theorem 3.11. So we see that in general the notion of isomorphism of bibundles is a finer equivalence relation, leading to more equivalence classes, than the notion of concordance of bibundles.

As a corollary of Theorem 3.11 above, we have the following result.

Theorem 3.12. *Let (H, G) be a crossed module. Then the bibundle $EG \times H \rightarrow EG \times_G H$ is a classifying bibundle in the sense that there is an isomorphism*

$$\pi_0 \mathbf{Bibun}_{(H, G)}(M) \cong [M, EG \times_G H]_{H/t(G)}$$

for any paracompact space M , which is induced by sending the isomorphism class of a bibundle P on M to the homotopy class of the map $\Gamma: M \rightarrow EG \times_G H$ over $H/t(G)$.

Proof. This follows because $EG \times H \rightarrow EG \times_G H$ is a bibundle and $EG \times H$ is contractible as a space over H . \square

3.6. Group structure on the classifying space. We will now show that there is a universal (H, G) bibundle $E(H, G) \rightarrow B(H, G)$ for which both $E(H, G)$ and $B(H, G)$ are topological groups and the projection map is a group homomorphism. Choose a model for the total space EG of the universal G bundle which can be

equipped with the structure of a topological group, containing G as a closed subgroup. Assume further that the action of H on G extends to an action of H on EG by automorphisms. Let us denote this extended action also by

$$\alpha: H \rightarrow \text{Aut}(EG).$$

Finally we assume that if $g \in G \subset EG$ then $\alpha(t(g))(e) = geg^{-1}$. In other words as well as G being a subgroup of EG we have (H, G) a crossed submodule of (H, EG) .

Example 3.9. Consider the crossed module $\Omega K \rightarrow PK$ with the action of PK on ΩK by conjugation. The space $E\Omega K = PK$ is a topological group under pointwise multiplication and the conjugation action on ΩK extends to the adjoint action of PK on $E\Omega K = PK$. This is also an action by automorphisms. Clearly if $g \in \Omega K$ and $e \in PK$ we have $\alpha(t(g))(e) = \alpha(g)(e) = geg^{-1}$.

More generally, the total space EG of the universal bundle can be constructed as the geometric realization of a certain simplicial space subject to a mild restriction on the topological group G (see for instance [9, 15]). It turns out that that EG carries a natural structure of a topological group, containing G as a closed subgroup. It is not hard to show (using the construction of EG given in [9]) that, since H acts by automorphisms on G , there is an induced action of H on EG and that moreover $\alpha(t(g))(e) = geg^{-1}$.

Given this extended action α we have a semi-direct product $EG \rtimes H$ with multiplication

$$(e, h)(e', h') = (e\alpha(h)(e'), hh').$$

We denote this group by $E(H, G)$. Notice that $(e, h)^{-1} = (\alpha(h)^{-1}(e^{-1}), h^{-1})$.

Notice also that the map $G \rightarrow E(H, G)$ defined by $g \mapsto (g^{-1}, t(g))$ is a homomorphism because

$$\begin{aligned} (g^{-1}, t(g))(k^{-1}, t(k)) &= (g^{-1}\alpha(t(g))(k^{-1}), t(g)t(k)) \\ &= (g^{-1}g(k^{-1})g^{-1}, t(g)t(k)) \\ &= (k^{-1}g^{-1}, t(gk)) \\ &= ((gk)^{-1}, t(gk)). \end{aligned}$$

Moreover the image of this map is actually a normal subgroup because

$$\begin{aligned} (3.11) \quad (e, h)(g^{-1}, t(g))(\alpha(h^{-1})(e^{-1}), h^{-1}) &= (e\alpha(h)(g^{-1}), ht(g))(\alpha(h^{-1})(e^{-1}), h^{-1}) \\ &= (e\alpha(h)(g^{-1})\alpha(ht(g))(\alpha(h^{-1})(e^{-1})), ht(g)h^{-1}) \\ &= ((\alpha(h)(g))^{-1}, t(\alpha(h)(g))). \end{aligned}$$

We denote the image of this homomorphism inside $E(H, G)$ by G and hence have the exact sequence of topological groups

$$(3.12) \quad G \rightarrow E(H, G) \rightarrow B(H, G).$$

In fact the map $E(H, G) \rightarrow B(H, G)$ admits local sections since it is obtained by pullback from the universal G bundle $EG \rightarrow BG$ via the projection $B(H, G) \rightarrow BG$ (see [3] for a proof of this). Notice that right action by G on $E(H, G)$ is

$$(e, h)g = (e, h)(g^{-1}, t(g)) = (e\alpha(h)(g^{-1}), ht(g)).$$

Example 3.2 shows that $E(H, G) \rightarrow B(H, G)$ is a $(G, E(H, G))$ bibundle, where the structure map $E(H, G) \rightarrow E(H, G)$ is the identity. Notice though that there is a morphism of crossed modules $(G, E(H, G)) \rightarrow (G, H)$, the homomorphism $E(H, G) \rightarrow H$ being the projection onto H in the semi-direct product. Therefore there is a natural extension of $E(H, G) \rightarrow B(H, G)$ to an (H, G) bibundle (see the discussion in subsection 2.2). The structure map $\Phi: E(H, G) \rightarrow H$ is then given by $\Phi(e, h) = h$.

Consider now the bijection $\hat{\chi}: EG \times H \rightarrow E(H, G)$ defined by

$$\begin{aligned} \hat{\chi}: EG \times H &\rightarrow E(H, G) \\ (e, h) &\mapsto (\alpha(h)(e^{-1}), h) \end{aligned}$$

and note that χ commutes with structure maps which are the projections onto H in both cases. To see that χ is a bibundle isomorphism we only have to check that it commutes with the right action of $g \in G$. We have

$$\begin{aligned} \hat{\chi}((e, h)g) &= \hat{\chi}(eg, ht(g)) \\ &= (\alpha(ht(g))(g^{-1}e^{-1}), ht(g)) \\ &= (\alpha(h)(\alpha(t(g))(g^{-1}e^{-1})), ht(g)) \\ &= (\alpha(h)(g(g^{-1}e^{-1})g^{-1}), ht(g)) \\ &= (\alpha(h)(e^{-1}g^{-1}), ht(g)) \\ &= (\alpha(h)(e^{-1})\alpha(h)(g^{-1}), ht(g)) \\ &= (\alpha(h)(e^{-1}), h)(g^{-1}, t(g)) \\ &= \hat{\chi}(e, h)g. \end{aligned}$$

It follows that $\hat{\chi}$ induces a map $\chi: EG \times_G H \rightarrow B(H, G)$ and that

$$(3.13) \quad \begin{array}{ccc} EG \times H & \xrightarrow{\hat{\chi}} & E(H, G) \\ \downarrow & & \downarrow \\ EG \times_G H & \xrightarrow{\chi} & B(H, G) \end{array}$$

is an isomorphism of bibundles.

The diagram (3.7) above now becomes a diagram of topological groups and continuous homomorphisms between them

$$\begin{array}{ccc} E(H, G) & \xrightarrow{\Psi} & H \\ \downarrow & & \downarrow \\ B(H, G) & \xrightarrow[\Phi]{} & H/t(G). \end{array}$$

If $P \rightarrow M$ is a bibundle we define $F: M \rightarrow B(H, G)$ by composing the function $\Gamma: M \rightarrow EG \times_G H$ with the function χ to obtain the composite diagram

$$\begin{array}{ccccc}
P & \xrightarrow{\hat{\Gamma}} & EG \times H & \xrightarrow{\hat{\chi}} & E(H, G) \\
\downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{\Gamma} & EG \times_G H & \xrightarrow{\chi} & B(H, G)
\end{array}$$

from which we obtain

$$\begin{array}{ccccc}
P & \xrightarrow{\hat{F}} & E(H, G) & \xrightarrow{\Psi} & H \\
\downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{F} & B(H, G) & \xrightarrow{\Phi} & H/t(G).
\end{array}$$

Clearly $E(H, G) \rightarrow B(H, G)$ is a classifying bibundle for (H, G) bibundles. Since $E(H, G) \rightarrow B(H, G)$ is a bibundle arising from the quotient of groups (3.12), it has a nice behaviour with respect to products of bibundles. More precisely we have the following results. The discussion in Example 3.6 gives us the following lemma.

Lemma 3.13.

- (1) *The product in the group $E(H, G)$ induces a morphism of bibundles $E(H, G) \otimes E(H, G) \rightarrow E(H, G)$ covering the product $B(H, G) \times B(H, G) \rightarrow B(H, G)$ in the group $B(H, G)$.*
- (2) *The inverse map $B(H, G) \rightarrow B(H, G)$ pulls back the bibundle $E(H, G) \rightarrow B(H, G)$ to its dual.*

From this we easily deduce the next proposition.

Proposition 3.14.

- (1) *Let $F_1, F_2: M \rightarrow B(H, G)$ and define $F_1 F_2: M \rightarrow B(H, G)$ to be the pointwise product. Then $(F_1 F_2)^*(E(H, G)) \simeq F_1^*(E(H, G)) \otimes F_2^*(E(H, G))$.*
- (2) *Let $F: M \rightarrow B(H, G)$ and denote by F^{-1} the pointwise inverse. Then $(F^{-1})^*(E(H, G)) \simeq (F^*(E(H, G)))^*$.*

As we have already observed, $E(H, G) \rightarrow B(H, G)$ is a universal bibundle in the sense of Theorem 3.11 and hence there is an isomorphism

$$(3.14) \quad [M, B(H, G)]_{H/t(G)} \cong \pi_0 \mathbf{Bibun}_{(H, G)}(M).$$

Since $B(H, G)$ is a topological group the set $[M, B(H, G)]_{H/t(G)}$ of homotopy classes of maps over $H/t(G)$ acquires a natural structure as a topological group. As we have remarked previously (see Lemma 3.2) $\pi_0 \mathbf{Bibun}_{(H, G)}(M)$ also has a natural structure of a group where the product $[P] \cdot [Q]$ is the isomorphism class of the bibundle $P \otimes Q$. Proposition 3.14 shows that the isomorphism (3.14) preserves products, since

$$[f^* E(H, G)] \cdot [g^* E(H, G)] = [(fg)^* E(H, G)].$$

It follows that (3.14) is an isomorphism of groups. We record this observation in the following theorem.

Theorem 3.15. *Let (H, G) be a crossed module. Then the bibundle $E(H, G) \rightarrow B(H, G)$ is a classifying bibundle which preserves group structures in the sense that there is an isomorphism of groups*

$$\pi_0 \mathbf{Bibun}_{(H, G)}(M) \cong [M, B(H, G)]_{H/t(G)}$$

for any paracompact space M .

3.7. (H, G) bibundle structures on a G principal bundle. Consider the commuting diagram

$$\begin{array}{ccc} EG \times H & \xrightarrow{\pi_{EG}} & EG \\ \downarrow & & \downarrow \\ EG \times_G H & \xrightarrow{\pi_{BG}} & BG \end{array}$$

where π_{EG} and π_{BG} are the natural projections. This is a morphism of G bundles and shows that π_{BG} is a classifying map for the G bundle $EG \times H \rightarrow EG \times_G H$. Using the isomorphism χ from the bibundle $EG \times H \rightarrow EG \times_G H$ to the bibundle $E(H, G) \rightarrow B(H, G)$, which is also a G bundle isomorphism, we see that $\pi: B(H, G) \rightarrow BG$ defined by $\pi = \pi_{BG} \circ \chi^{-1}$ is a classifying map for the G bundle $E(H, G) \rightarrow B(H, G)$. Note that $\pi: B(H, G) \rightarrow BG$, like π_{BG} , has fibre H .

It follows that if $P \rightarrow M$ is an (H, G) bibundle with classifying map $F: M \rightarrow B(H, G)$ then $\pi \circ F: M \rightarrow BG$ is a classifying map for the G bundle $P \rightarrow M$. Conversely if a G bundle $P \rightarrow M$ has a classifying map $f: M \rightarrow BG$ which lifts to a map $\hat{f}: M \rightarrow B(H, G)$ then it is isomorphic to $f^*(E(H, G))$ and thus admits an (H, G) bibundle structure. Hence a G bundle $P \rightarrow M$ admits an (H, G) bibundle structure if and only if it has a classifying map $M \rightarrow BG$ which lifts to $M \rightarrow B(H, G)$. Thus we have the following proposition.

Proposition 3.16. *If H is contractible then every G bundle admits an (H, G) bibundle structure.*

Example 3.10. Consider a ΩK bundle $P \rightarrow M$. Then as PK is contractible we can always lift a map $M \rightarrow B\Omega K$ to $B(PK, \Omega K)$ and every ΩK admits a $(PK, \Omega K)$ bibundle structure. Recall that a $(PK, \Omega K)$ is also an ΩK bibundle so every ΩK bundle admits an ΩK bibundle structure.

3.8. Loop groups. We have seen that the existence of bibundles that are not abelian relates to the size of $\text{Out}(G)$ and we have commented that we are therefore interested in groups that have large outer automorphism groups. One example is the group $G = \Omega K$ of based loops in a compact Lie group K . There are a number of possible groups H and homomorphisms making $\Omega K \rightarrow H$ into a crossed module so for the moment we will make some general comments, before looking at some specific examples.

As we remarked above we have $E\Omega K = PK$ and $B\Omega K = K$ with the projection $PK \rightarrow K$ being evaluation at 1. We have that $EG \times H = PK \times H$ where the action of ΩK is $(e, h)k = (ek, ht(k))$ and that $E(H, \Omega K) = PK \times H$ with the product $(e, h)(e', h') = (e\alpha(h)(e'), hh')$ and the action of $k \in \Omega K$ being $(e, h)k = (e\alpha(h)(k^{-1}), ht(k))$. The projection from $B(H, \Omega K)$ to $B\Omega K = K$ is given by $[e, h] \mapsto \alpha(h)^{-1}(e^{-1})(1)$. The type map $B(H, \Omega K) \rightarrow H/t(\Omega K)$ is just projection of (e, h) to the coset of h in $H/t(\Omega K)$.

Example 3.11. Let $H = PK$ with the action $\alpha(h)(e) = heh^{-1}$ using pointwise multiplication of paths. Then $E(PK, \Omega K) = PK \times PK$ with the multiplication

$$(e, h)(\bar{e}, \bar{h}) = (eh\bar{e}h^{-1}, h\bar{h}).$$

We can identify this with the usual product $PK \times PK$ by the isomorphism $(e, h) \mapsto (eh, h)$. The subgroup ΩK of all (k^{-1}, k) becomes the subgroup of all $(1, k)$ or $\{1\} \times \Omega K$. So we have $E(PK, \Omega K) = PK \times PK$ and $B(PK, \Omega K) = PK \times K$. The projection of a pair $(e, k) \in PK \times K$ to $B\Omega K = K$ can be calculated by first reversing the isomorphism above to send it to (ek^{-1}, k) and then mapping this to $\alpha(k)^{-1}((ek^{-1})^{-1})(1) = e(1)k$. Notice that any map $f: M \rightarrow K$ can be lifted to $F: M \rightarrow PK \times K$ by taking $F(m) = (1, f(m))$.

In this example $H/t(\Omega K) = PK/\Omega K = K$ and the type map $B(PK, \Omega K) = PK \times K \rightarrow K$ is just projection onto the second factor. It follows that we have the exact sequence of groups

$$1 \rightarrow PK \rightarrow B(PK, \Omega K) \rightarrow K \rightarrow 1.$$

This is an instance of an exact sequence of groups

$$1 \rightarrow EG/G_1 \rightarrow B(H, G) \rightarrow H/t(G) \rightarrow 1$$

which exists for any crossed module (H, G) .

Using these observations we illustrate how the set of homotopy classes of maps $[M, B(H, G)]$ can fail to be isomorphic to the set of isomorphism classes of (H, G) -bibundles, as mentioned earlier.

Example 3.12. Let K be a compact connected, non-trivial Lie group and let $G = \Omega K$. Then we have seen in Example 3.11 above that $B(\Omega K, PK)$ is isomorphic to the group $PK \times K$ and the homomorphism $\Phi: B(\Omega K, PK) \rightarrow K$ is the natural projection. Clearly we can find homotopic maps $F_0, F_1: M \rightarrow PK \times K$ such that ΦF_0 and ΦF_1 are not equal. Hence the ΩK bibundles induced by pull back with F_0 and F_1 are not isomorphic.

We leave it for the interested reader to consider some of the other crossed modules $(H, \Omega K)$ in the following examples.

Example 3.13. The group $\text{Diff}_0([0, 1])$ of diffeomorphisms fixing 0 and 1 acts on ΩK so we can take H equal to the semi-direct product $PK \rtimes \text{Diff}_0(S^1)$.

Example 3.14. Replace PK by the group of smooth maps from $[0, 1]$ to $\text{Aut}(K)$ acting pointwise. As automorphisms fix the identity there is no need to impose a condition on the map at the endpoints.

4. CONCLUSION

While bibundles are of independent interest one motivation for discussing them is the notion of an (H, G) bibundle gerbe [1]. We will indicate here briefly how our approach will apply in this case. A complete discussion will appear in [13].

We assume the reader is familiar with abelian bundle gerbes [12]. First we have the analogue of the definition of an abelian bundle gerbe.

Definition 4.1 (c.f. [1]). An (H, G) *bibundle gerbe* on M , or just bibundle gerbe when the crossed module (H, G) is understood, consists of a pair (P, Y) where $\pi: Y \rightarrow M$ is a surjective submersion and $P \rightarrow Y^{[2]}$ is an (H, G) bibundle equipped

with a bibundle gerbe product. This is a bibundle map which on fibres takes the form

$$P_{(y_1, y_2)} \otimes P_{(y_2, y_3)} \rightarrow P_{(y_1, y_3)}$$

for $(y_1, y_2, y_3) \in Y^{[3]}$. The bibundle gerbe product is required to be associative in the usual sense.

Fundamental to the theory of bibundle gerbes is the notion of stable isomorphism. Before we give the definition, observe that if (P, Y) is a bibundle gerbe and R is a bibundle on Y , then we can construct a new bibundle gerbe $(\pi_2^* R^* \otimes P \otimes \pi_1^* R, Y)$ on M with bibundle gerbe product given fibrewise by using the bibundle gerbe product on P and contraction as

$$R_{y_1}^* \otimes P_{(y_1, y_2)} \otimes R_{y_2} \otimes R_{y_2}^* \otimes P_{(y_2, y_3)} \otimes R_{y_3} \rightarrow R_{y_1}^* \otimes P_{(y_1, y_3)} \otimes R_{y_3}.$$

With this construction in hand we can make the following definition, which is closely related to Definition 12 in [1].

Definition 4.2. Let (P, Y) and (P', Y') be bibundle gerbes on M . We say that P is *stably isomorphic* to P' if there exists a bibundle R on $Y \times_M Y'$ together with an isomorphism of bibundle gerbes

$$(4.1) \quad \pi_2^* R^* \otimes P \otimes \pi_1^* R \cong P'$$

where we have suppressed the projections $(Y \times_M Y')^{[2]} \rightarrow Y^{[2]}$ and $(Y \times_M Y')^{[2]} \rightarrow (Y')^{[2]}$ which are used to pullback P and P' respectively.

Finally the type of an (H, G) bibundle gerbe is defined as follows. Let (P, Y) be an (H, G) bibundle gerbe over M . Then $P \rightarrow Y^{[2]}$ has a type map $Y^{[2]} \rightarrow H/t(G)$. The existence of the bundle gerbe multiplication and its associativity means that

$$(4.2) \quad \phi(y_1, y_2)\phi(y_2, y_3) = \phi(y_1, y_3)$$

for all $(y_1, y_2, y_3) \in Y^{[3]}$. Recall [5] that a K *prebundle* over M consists of a submersion $Y \rightarrow M$ and a map $k: Y^{[2]} \rightarrow K$ satisfying a cocycle equation analogous to (4.2) above. Every K prebundle over M determines a principal K -bundle over M and conversely.

It follows that every (H, G) bibundle gerbe (P, Y) over M defines an $H/t(G)$ prebundle (ϕ, Y) over M and hence a principal $H/t(G)$ bundle over M . We call this the *type* of the bibundle gerbe. In the sheaf theoretic setting, this pre-bundle is known as the *band* of the gerbe.

Example 4.1. A Jandl bundle gerbe [14] consists of a $(\mathbb{Z}_2, U(1))$ bundle gerbe (P, Y) over M . The type of (P, Y) is a \mathbb{Z}_2 prebundle over M and the induced \mathbb{Z}_2 -bundle is called the *orientation bundle* of the Jandl bundle gerbe.

We will discuss all of these notions in more detail in [13].

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